Pre-class Warm-up!!!

How many methods have we learned so far to solve differential equations?

a. 0

b. 1

c. 2

d. 3

e. ≥ 4

Section 1.6: Substitution methods and exact equations

We learn to recognize several kinds of differential equation:

- homogeneous equations
- Bernoulli equations
- Equations where some special substitution works
- Higher order equations where the order can be reduced
- Exact equations

The first 4 of these can be dealt with by introducing a new variable v dependent on both y and x: $v = g(x,y)$, where we can also write $y = h(x,v)$. We then get dy/dx in terms of dv/dx.

Exact equations are different!

Compare the methods we already learned:

- y' = function of x
- separate the variables
- first order linear equations

substitute $v = g(x,y)$

Homogeneous equations

Page 74 question 8 (like page 60 example 2)
Find the general solution to
2xy + x^2 y′ = y^2

$$x^2\frac{dy}{dx} + 2xy = y^2$$

A homogeneous equation is one of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

In this case : divide both sides by x^2

$$\frac{dy}{dx} + 2\frac{xy}{x^2} = \frac{y^2}{x^2}$$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 - 2\left(\frac{y}{x}\right)$$

Method: Put $v = \frac{y}{x}$

so $y = vx$, $\frac{dy}{dx} = \frac{d(vx)}{dx}$

$$= v + x\frac{dv}{dx}$$

Substitute.

$$v + x\frac{dv}{dx} = v^2 - 2v$$

$$x\frac{dv}{dx} = v^2 - 3v$$

$$\frac{dv}{dx} = \frac{v^2 - 3v}{x}$$

We can solve this by:

- integrating y′ = function of x
- ✓ separating the variables
- as a first order linear equation

Page 74 question 8 (like page 60 example 2)
Find the general solution to
2xy + x^2 y′ = y^2

Summary: $v = \frac{y}{x}$ $\quad x\frac{dv}{dx} = v^2 - 3v$

Solve this by separating the variables.

$$\int \frac{dv}{v^2-3v} = \int \frac{dx}{x}$$

Partial fractions (calculation not shown)

$$\int \frac{1}{3}\left(\frac{1}{v-3} - \frac{1}{v}\right) dv = \int \frac{dx}{x}$$

$$\frac{1}{3}\left(\ln|v-3| - \ln|v|\right) = \ln x + C$$

$$\ln\left(\frac{v-3}{v}\right)^{\frac{1}{3}} = \ln x + C$$

$$\left(\frac{v-3}{v}\right)^{\frac{1}{3}} = e^{\ln x + C} = Bx \text{ where } B = e^C$$

$$\frac{v-3}{v} = B^3x^3$$

$$v - 3 = B^3x^3v$$

$$v(1 - B^3x^3) = 3$$

$$v = \frac{3}{1-B^3x^3} = \frac{y}{x}$$

$$y = \frac{3x}{1-B^3x^3}$$

Bernoulli equations

1.6 question 23.
Find the general solution to
xy′ + 6y = 3x y^(4/3)

$x\frac{dy}{dx} + 6y = 3x\,y^{4/3}$ is almost a first order linear equation.

Form of a Bernoulli equation:

$y' + P(x)\,y = Q(x)\,y^n$ extra ingredient

Method: Substitute $v = y^{1-n}$

In this case: $n = \frac{4}{3}$, put $v = y^{1-\frac{4}{3}} = y^{-\frac{1}{3}}$

$y = v^{-3}$, $\frac{dy}{dx} = \frac{d\,v^{-3}}{dx} = \frac{d\,v^{-3}}{dv}\cdot\frac{dv}{dx}$

$= -3v^{-4}\frac{dv}{dx}$

Substitute: $x(-3v^{-4})\frac{dv}{dx} + 6v^{-3} = 3x\,v^{-4}$

Multiply by $-\frac{v^4}{3x}$: $\frac{dv}{dy} - \frac{2v}{x} = -1$

$\frac{dv}{dx} - \frac{2v}{x} = -1$ is first order linear.

Integrating factor: $e^{\int -\frac{2}{x}dx} = e^{-2\ln x}$

$= e^{\ln x^{-2}} = x^{-2}$

$\frac{1}{x^2}\frac{dv}{dx} - \frac{2v}{x^3} = \frac{-1}{x^2}$

$\frac{d}{dx}\left(\frac{v}{x^2}\right) = -\frac{1}{x^2}$, $\frac{v}{x^2} = \frac{1}{x} + C$

$v = x + Cx^2 = y^{-\frac{1}{3}}$

$y = (x + Cx^2)^{-3}$

Special substitutions

Questions from Section 1.6: solve the equations

16. $y' = \sqrt{(x+y+1)}$

28. $x e^y y' = 2(e^y + x^3 e^{2x})$

Solution to 16. We are going to make a substitution v = something. What should it be?

a. $v = e^{\text{something}}$

b. $v = y^{\text{something}}$

c. $v = \frac{y}{x}$

d. $v = x+y+1$

e. $v = \sqrt{x+y+1}$ ✓

$x+y+1 = v^2, \quad y = v^2 - x - 1$

$\frac{dy}{dx} = 2v\frac{dv}{dx} - 1$. Substitute:

$2v\frac{dv}{dx} - 1 = v$

$2v\frac{dv}{dx} = v+1$

$\frac{dv}{dx} = \frac{v+1}{2v}$ Separate the variables.

$\int \frac{2v\,dv}{v+1} = \int dx$

$\int \frac{2v+2-2}{v+1}dv = \int\left(2 - \frac{2}{v+1}\right)dv = 2v - 2\ln(v+1)$

$= \int dx = x + C$

$2\sqrt{x+y+1} - 2\ln\left(1+\sqrt{x+y+1}\right) = x + C$

gives a solution implicitly.

Questions from Section 1.6: solve the equation

28. $x e^y y' = 2\left(e^y + x^3 e^{2x}\right)$

What substitution should we make?

a. $v = e^y + x^3 e^{2x}$

b. $v = e^y$

c. $v = x e^y$

d. $v = x^3 e^{2x}$

e. $v =$

Try $v = e^y$. $y = \ln v$

$$\frac{dy}{dx} = \frac{1}{v}\frac{dv}{dx}$$ Substitute.

$$x v \frac{1}{v}\frac{dv}{dx} = 2\left(v + x^3 e^{2x}\right)$$

Divide by x:

$$\frac{dv}{dx} = 2\left(\frac{v}{x} + x^2 e^{2x}\right).$$

$$\frac{dv}{dx} - \frac{2v}{x} = 2x^2 e^{2x}$$

We can solve this

a. by separating the variables

✓ b. as a first order linear equation

c. by making another substitution

Reducing the order of a differential equation

Section 1.6 question 44. Solve y y'' = (y')^2

Solution: it's significant that there is no term in x in the equation.

Substitute: $v = \frac{dy}{dx}$

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$$

Substitute:

$$yv\frac{dv}{dy} = v^2.$$

Divide by yv.

$$\frac{dv}{dy} - \frac{1}{y}v = 0$$

$$\text{I.F. } e^{\int -\frac{1}{y}dy} = e^{-\ln y} = \frac{1}{y}$$

$$\frac{1}{y}\frac{dv}{dy} - \frac{1}{y^2}v = \frac{d}{dy}\left(\frac{v}{y}\right) = 0$$

$$\frac{v}{y} = C, \quad v = Cy = \frac{dy}{dx}$$

- We can solve this

a. by separating the variables
b. as a first order linear equation
c. by making another substitution

$y = Be^{Cx}$ with two constants: B, C

Section 1.6 question 46: Solve $x\,y'' + y' = 4x$

This time it's significant there is no term in y.

Put $v = \frac{dy}{dx} = y'$

$$y'' = \frac{dv}{dx}$$

Substitute: $x\frac{dv}{dx} + v = 4x$.

$$\frac{dv}{dx} + \frac{1}{x}v = 4$$

We can solve this

a. by separating the variables

✓ b. as a first order linear equation

c. by making another substitution

Then solve the equation for v that arises.

Pre-class Warm-up!!!

How would you solve the following equation?

2xy + x^2 y′ = y^2

a. by integrating y′ = function of x

b. separate the variables

c. as a first order linear equation $v = \frac{y}{x}$

✓ d. as a homogeneous equation $y' + 2v = v^2$

✓ e. as a Bernoulli equation $y' + \frac{2}{x} y = \frac{y^2}{x^2}$

f. make a special substitution

g. reduce the order

What about xy^2 + 3y^2 - x^2y′ = 0

xy′ + 2y = 6x^2 √y

y′ = √(x+y)

Exact equations

Question: Solve $2xy\frac{dy}{dx} + y^2 = 10x$

Solution: This equation is

$$\frac{d}{dx}\left(xy^2\right) = 10x$$

$$xy^2 = \int 10x\,dx = 5x^2 + C$$

$$y^2 = 5x + \frac{C}{x}$$

$$y = \sqrt{5x + \frac{C}{x}}$$

How do we find a function f(x,y) so that

$$\frac{d}{dx}f = 2xy\frac{dy}{dx} + y^2 \quad ?$$

Notice that $\frac{df}{dx} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial x}$

We solve $f_y = 2xy \quad f_x = y^2$

Thus $f = xy^2 + g(x) \qquad f = xy^2 + h(y)$

$f = xy^2$ works.

xy^2 is a potential function for the vector field

$$\begin{bmatrix} y^2 \\ 2xy \end{bmatrix} = \nabla(xy^2)$$

It can look better to write the equation in differential form:

2xy dy + y^2 dx = 10 dx

d(xy^2) = 10 dx

$(2x^2 y + 1)\, dy/dx + 2xy^2 = 2x$

The left side is $\frac{df}{dx}$ where f is

a. $x^2 y + y$

b. $x^2 y^2 + 1$

✓ c. $x^2 y^2 + y$

d. $x y^2 + 2x^2 y$

e. None of the above.